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Analogous Riemannian Metric Description of Gravitational and Electromagnetic Interactions

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Abstract

It is shown that, if (a) the gravitational field is represented by the metric tensor of a Riemann space, (b) the geodesic hypothesis is admitted, (c) it is assumed that, for a particle moving in a scalar gravitational field, this last postulate must give, by approximation, the Hamilton principle for this particle in special relativity, then a metric tensor is obtained with some interesting properties (no assumptions are made about field equations). The geodesic hypothesis, with the Lorentz transformation of this metric tensor, gives, by approximation, the Hamilton principle, with the Lagrangian corresponding in special relativity to a particle in a vector field. Moreover, the equations of motion in a Riemann space, as they follow from the geodesic postulate, in terms of associated coordinates, give, by approximation, an expression in complete analogy to that of the Lorentz force. Hence, the vector-field theory of gravitation of the Maxwell–Lorentz kind (as electrodynamics, because of the formal analogy between the two theories) is obtained as a weak field approximation of a description of gravitational (electromagnetic) interaction by a metric tensor in a Riemann space (except the field equations, an issue that we do not touch, so far).

Since the formulation of the special theory of relativity, a Lorentz-covariant theory of gravitation has been sought by several people, while Einstein, instead, founded the general theory of relativity. Nevertheless, investigations have continued in order to find a description of gravitational interaction within the framework of special relativity, hopefully able to give the same results as those given by general relativity for at least the three experimental facts that represent the empirical confirmation of Einstein's general theory. In this sense, different formulations have been proposed, with scalar, vector, or tensor potentials, and combinations thereof, in a manifestly Lorentz-covariant field-theoretic context (Robertson and Noonan, 1968).

Among the theories using a vector potential, one similar to the Maxwell–Lorentz electrodynamics has been studied. This theory fails, however, for it gives one sixth the observed value for the precession of the perihelion of Mercury (Bergman, 1942). Notwithstanding this fact, investigations have continued up to recent years along these lines (Carstoiu, 1969a–c; Leiby, 1972; Majernik, 1971, 1972a, b; Spieweck, 1971).

It is the purpose of this work to show that this particular vector-field theory of gravitation can be obtained as a weak-field approximation of a description of gravitational interaction by means of the metric tensor of a Riemannian space; hence, the same conclusion holds for electrodynamics, because of the formal analogy between both theories. We do not dare to say, however, that it is an approximation of some full-fledged theory of gravitation in a Riemannian space, for we shall not touch on the issue of field equations in the present context. In other words, of the two fundamental postulates of general relativity (Synge, 1966; Bunge, 1967), i.e., representation of gravitation by means of a Riemannian metric tensor and assumption of a particular system of equations of field, we will use the first but not the second. Besides, we shall admit the geodesic hypothesis for a particle moving in a given gravitational field.

We here recall the equations of motion in a Riemann space, as they follow from the geodesic hypothesis. We also stress the fact that, in the weak-field approximation, an expression formally identical to the Lorentz force obtains. The fact that one gets a rotational term is well known, which is usually related to the Coriolis force.

It can be shown (Arzelies, 1961) that the equations of motion of a free particle in a Riemannian space, in terms of the associated space and time coordinates, are

$$\begin{aligned}\frac{Dp_\alpha}{Dt} = & -m\frac{\partial}{\partial t}(g_\alpha g_\beta v^\beta + icg_{\alpha 4}) - (g_\alpha g_\beta v^\beta + icg_{\alpha 4})\frac{\partial m}{\partial t} \\ & + \frac{mv^\delta}{\kappa}(g_\alpha g_\beta v^\beta + icg_{\alpha 4})\frac{\partial \kappa}{\partial x^\delta} - mv^\delta \frac{\partial}{\partial x^\delta}(g_\alpha g_\beta v^\beta + icg_{\alpha 4}) \\ & + \frac{mv^\beta v^\delta}{2}\frac{\partial(g_\beta g_\delta)}{\partial x^\alpha} + icmv^\beta \frac{\partial g_{\beta 4}}{\partial x^\alpha} - \frac{mc^2}{2}\frac{\partial g_{44}}{\partial x^\alpha}\end{aligned} \tag{1}$$

where D/Dt denotes the absolute derivative, and where

$$p_\alpha = mv_\alpha, \qquad m = m_0\kappa^{-1}, \qquad g_\alpha = g_{\alpha 4}/(g_{44})^{1/2}$$
$$\kappa = \left[\left((g_{44})^{1/2} - \frac{ig_\alpha v^\alpha}{c}\right)^2 - \frac{v^2}{c^2}\right]^{1/2} \tag{2}$$

with $\alpha, \beta, \delta = 1, 2, 3$.

For weak fields, the metric tensor is assumed to be $g_{ii} = 1 + \epsilon f_i$ and $g_{ij} = \epsilon q_{ij}$ ($i \neq j; i, j = 1, 2, 3, 4$), where f_i and q_{ij} are functions of the coordinates. Hence, we get, after neglecting second-order terms in ϵ, while not assuming any condition on the velocity (contrary to Arzelies, 1961)

$$\frac{Dp_\alpha}{Dt} = -\frac{mc^2}{2}\frac{\partial g_{44}}{\partial x^\alpha} - imc\frac{\partial g_{\alpha 4}}{\partial t} + imcv^\beta\left(\frac{\partial g_{\beta 4}}{\partial x^\alpha} - \frac{\partial g_{\alpha 4}}{\partial x^\beta}\right) \tag{3}$$

For future reference, we here mention the fact that this result is also valid in the case $g_{ij} = \epsilon^a q$, when terms of order $a + 1$ are neglected, with $a > 1$.

The right-hand member of equation (3) is called the force acting on the particle. Should the $g_{\alpha 4}$'s be the components of a vector, say $\mathbf{g}_4$ (which is not in general the case), we could write

$$\mathbf{F} = -\frac{mc^2}{2}\nabla g_{44} - imc\frac{\partial \mathbf{g}_4}{\partial t} + imc\mathbf{v} \times (\nabla \times \mathbf{g}_4)$$

or, with a change of notation,

$$\mathbf{H} = -\nabla U - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \tag{4}$$

where

$$m\mathbf{H} = \mathbf{F}, \qquad \nabla U = \tfrac{1}{2}c^2 \nabla g_{44}$$

$$\frac{\partial \mathbf{A}}{\partial u} = ic\frac{\partial \mathbf{g}_4}{\partial u} \qquad (u = x, y, z, t)$$

Besides the fact that, in general, $g_{\alpha 4}$ is not a vector, there is another difficulty in establishing an analogy with electrodynamics. When the values of g_{ij} given by the general theory of relativity for the field produced by a particle with uniform motion relative to an inertial system are used, one gets a factor 4 in $\mathbf{A}(= ic\mathbf{g}_4)$, in comparison to that of the Lorentz-covariant theory of the Maxwell–Lorentz kind. This result has been interpreted assuming that the velocity of the gravitational waves is half that of the light (Forward, 1961).

We now study the possibility of having an analogy from the standpoint of the variational principle. We shall calculate the components g_{ij} of the Riemannian metric such that the geodesic hypothesis will give us, by approximation, the variational principle for a particle in a scalar gravitational field in the context of the special theory of relativity. We consider the gravitational field of a particle at the origin; so that, because of the symmetry of the problem, only the diagonal components g_{ii} of the metric tensor will be different from zero. Moreover, the three spatial components of the metric will be equal. Afterwards, we calculate the metric tensor corresponding to the field of a uniformly moving particle; this we do by means of a Lorentz transformation of the initial metric tensor. In this way, we succeed in obtaining an expression for the force in complete analogy to that of the Maxwell–Lorentz theory of electrodynamics.

In the special theory of relativity, the Lagrangian of a particle in a gravitational field U is given by

$$L = -m_0c^2(1 - \beta^2)^{1/2} - m_0 U \tag{5}$$

Hence, the principle of Hamilton states, in this case,

$$\delta \int [-m_0c^2(1 - \beta^2)^{1/2} - m_0 U]\, dt = 0 \tag{6}$$

Since, on the other hand, the geodesic hypothesis in a Riemann space gives, for the motion of a free particle,

$$\delta \int ds = 0$$

we must have

$$ds \propto [-(1-\beta^2)^{1/2} - U/c^2]c\,dt$$

i.e.,

$$ds^2 \propto \left[(1-\beta^2) + \frac{U^2}{c^4} + 2\frac{U}{c^2}(1-\beta^2)^{1/2}\right]c^2 dt^2$$

Now, assuming $\beta^2 \sim \epsilon^{1/2}$ and neglecting terms of second order in ϵ, while keeping those of order 3/2, we get

$$ds^2 \propto \left(1 - \frac{v^2}{c^2} + \frac{2U}{c^2} - \frac{U}{c^2}\frac{v^2}{c^2}\right)c^2\,dt^2$$

which may be factorized as

$$ds^2 \propto -\left(1 + \frac{U}{c^2}\right)v^2\,dt^2 + \left(1 + \frac{2U}{c^2}\right)c^2\,dt^2$$

or, with $v^2 = (d\mathbf{r}/dt)^2$ and $dx_4 = ic\,dt$

$$ds^2 \propto \left(1 + \frac{U}{c^2}\right)d\mathbf{r}^2 + \left(1 + \frac{2U}{c^2}\right)dx_4{}^2 \qquad (7)$$

One can thus obtain a variational principle for a not-too-slow particle (i.e., $\beta^2 \sim \epsilon^{1/2}$), in the context of the special theory of relativity, as an approximate result of the geodesic principle in a Riemann space with the following metric tensor (neglecting second-order terms in ϵ): $g_{\alpha\alpha} = 1 + U/c^2$, $\alpha = 1, 2, 3$; $g_{44} = 1 + 2U/c^2$; $g_{ij} = 0$, $i \neq j$, $i, j = 1, 2, 3, 4$.

Now, we shall do a Lorentz transformation of this metric tensor in order to get that of the gravitational field of a particle with uniform motion (relative to the first, assumed inertial, frame of reference). If g'_{ij} are the components of the metric tensor in the frame of reference S', which is moving with velocity $\mathbf{V}$ relative to the initial frame of reference S, we have

$$g'_{ij} = a_i{}^k a_j{}^l g_{kl}$$

But, in this case, $g_{kl} = \delta_k{}^l g_{ll}$, hence

$$g'_{ij} = a_i{}^l a_j{}^l g_{ll}$$

With our values for the metric tensor, we get

$$g'_{ij} = a_i{}^\alpha a_j{}^\alpha \left(1 + \frac{U}{c^2}\right) + a_i{}^4 a_j{}^4 \left(1 + \frac{2U}{c^2}\right)$$

and, factorizing,

$$g'_{ij} = a_i{}^l a_j{}^l \left(1 + \frac{U}{c^2}\right) + a_i{}^4 a_j{}^4 \frac{U}{c^2}$$

i.e.,

$$g'_{ij} = \delta_i{}^j \left(1 + \frac{U}{c^2}\right) + a_i{}^4 a_j{}^4 \frac{U}{c^2} \tag{8}$$

Then, with $\boldsymbol{\beta}' = \mathbf{V}/c$ and $\gamma = (1 - \beta'^2)^{1/2}$, the components of the metric tensor are

$$\begin{aligned} g'_{\alpha\alpha} &= 1 + \frac{U}{c^2} + (i\gamma\beta'_\alpha)\frac{U}{c^2} \\ g'_{\sigma\rho} &= (i\gamma\beta'_\sigma)(i\gamma\beta'_\rho)\frac{U}{c^2} \\ g'_{\alpha 4} &= (i\gamma\beta'_\alpha)\gamma\frac{U}{c^2} \\ g'_{44} &= 1 + \frac{U}{c^2} + \gamma^2\frac{U}{c^2} \end{aligned} \tag{9}$$

Thus, assuming $\beta'^2 = V^2/c^2 \sim \epsilon$ and neglecting second-order terms in ϵ, we get

$$\begin{aligned} g'_{\alpha\alpha} &= 1 + \frac{U}{c^2}, \qquad g'_{44} = 1 + \frac{2U}{c^2} \\ g'_{\alpha 4} &= i\beta_\alpha \frac{U}{c^2}, \qquad g'_{\sigma\rho} = 0 \qquad (\sigma \neq \rho) \end{aligned} \tag{10}$$

With these values for the metric tensor one has an expression for the force which is identical to that of the Maxwell-Lorentz electrodynamics [including potentials as those of Lienard and Wiechert, if the "inverse of the distance" potential is assumed in S; note that in this case $g'_{\alpha 4} \sim \epsilon^{3/2}$ and see remark after equation (3)].

It can be shown that this metric tensor is compatible with the Lagrangian associated with a particle in a scalar field U and a vector field $\mathbf{A}$, i.e., with

$$L = -m_0 c^2 (1 - \beta^2)^{1/2} - m_0 U + m_0 \mathbf{v} \cdot \mathbf{A} \tag{11}$$

One can do a calculation similar to that already presented, which gives the metric tensor from this Lagrangian. If the field is produced by a particle with velocity $\mathbf{u} = -\mathbf{V}$, we assume $|\mathbf{u}/c| \sim \epsilon^{1/2}$ (then $|\mathbf{A}/c| \sim \epsilon^{3/2}$, because $U/c^2 \sim \epsilon$), $\beta^2 = v^2/c^2 \sim \epsilon^{1/2}$, as before, and we neglect second-order terms in ϵ (but we keep terms of the order 7/4 in ϵ; this implies that, if $\epsilon \sim 10^{-8}$, as is the case of the solar field in Mercury, there is a difference of two orders of magnitude between the terms neglected and the smallest terms kept).

It can also be shown that this metric tensor gives, as an approximation, the respective Lagrangian of special relativity. The geodesic hypothesis can be written (Moller, 1952)

$$\delta \int \left(-g_{lk} \frac{dx^l}{d\lambda} \frac{dx^k}{d\lambda}\right)^{1/2} d\lambda = 0 \tag{12}$$

Now, since the integrand is an homogeneous function of first degree in the four variables $dx^i/d\lambda$, equation (12) is equivalent to a variational problem with only three dependent variables x^l and with t as the independent variable (Courant and Hilbert, I, p. 196, cited in Moller, 1952). We get, for all variations δx^l that vanish for $t = t_1$ and $t = t_2$,

$$\delta \int F(\dot{x}^l, x^l, t)\, dt = 0$$

The function $F = ds/dt$ must be proportional to the Lagrangian in order for the geodesic hypothesis to give Hamilton's principle. Then, we write

$$L(\dot{x}^l, x^l, t)\, dt = -m_0 c \frac{ds}{dt}$$

or, from (2), since $ds/dt = c\kappa$,

$$L = -m_0 c^2 \left\{ \left[(g_{44})^{1/2} - \frac{i g_{\alpha} v^{\alpha}}{c} \right]^2 - \frac{v^2}{c^2} \right\}^{1/2}$$

Thus, in associated coordinates, where $v^2 = \gamma_{\alpha\alpha}\, (dx^{\alpha}/dt)^2$ and $\gamma_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha} g_{\beta}$, we can write

$$L = -m_0 c^2 \left[g_{44} \left(1 - \frac{i g_{\alpha 4} v^{\alpha}}{c g_{44}} \right)^2 - g_{\alpha\alpha} \beta_E{}^2 + g_{\alpha} g_{\alpha} \left(\frac{dx^{\alpha}}{dt} \right)^2 \right]^{1/2}$$

with

$$\beta_E{}^2 = \sum_{\alpha=1}^{3} (dx^{\alpha}/dt)^2$$

Neglecting quadratic terms in $g_{\alpha 4}$, i.e., terms of order equal to or higher than the second in ϵ (but keeping those of order 7/4[1]), we get

$$L = -m_0 c^2 \left(g_{44} - 2 \frac{i g_{\alpha 4} v^{\alpha}}{c} - g_{\alpha\alpha} \beta_E{}^2 \right)^{1/2}$$

For our metric tensor, with $g_{\alpha 4} = -iA_{\alpha}/c$, we have

$$L = -m_0 c^2 \left[1 + \frac{2U}{c^2} - \frac{2A_{\alpha} v^{\alpha}}{c^2} - \left(1 + \frac{U}{c^2} \right) \beta_E{}^2 \right]^{1/2}$$

So, reordering and factorizing, we have

$$L = -m_0 c^2 (1 - \beta_E{}^2)^{1/2} \left[1 + \frac{2U}{c^2} \frac{1 - \frac{1}{2}\beta_E{}^2}{1 - \beta_E{}^2} - \frac{2A_{\alpha} v^{\alpha}}{c^2 (1 - \beta_E{}^2)} \right]^{1/2}$$

Approximating the second square root, neglecting terms of second or higher order in ϵ, we get

$$L = -m_0 c^2 \left[(1 - \beta_E{}^2)^{1/2} + \frac{U}{c^2} \frac{1 - \frac{1}{2}\beta_E{}^2}{(1 - \beta_E{}^2)^{1/2}} - \frac{A_{\alpha} v^{\alpha}}{c^2 (1 - \beta_E{}^2)^{1/2}} \right]$$

[1] Remember that in our approximation $\beta_E{}^2 \sim \epsilon^{1/2}$, $U/c^2 \sim \epsilon$, and $g_{\alpha 4} \sim \epsilon^{3/2}$.

and again neglecting terms of second or higher order in ϵ, the following obtains

$$L = -m_0 c^2 (1 - \beta_E{}^2)^{1/2} - m_0 U + m_0 \mathbf{v} \cdot \mathbf{A}$$

These results, which give a Lorentz-covariant theory of motion in a gravitational field, can also be used to give a description of electrodynamics as a metric tensor in a Riemann space (of course, in this case the metric depends on the ratio e/m of the test particle); clearly, one has the potentials and the Lorentz force, but not the field equation. This fact is related to the hypothesis we have used. Indeed, we have assumed the geodesic motion and the representation of the field by the metric tensor, but we have not used the Einstein field equations. It is perhaps of some interest to note the fact that here the electromagnetic field would be represented by a symmetric tensor, and not necessarily by an antisymmetric one, as is usually assumed. Surely, we are concerned with two different fields: The second one (i.e., the antisymmetric) is the ordinary electromagnetic field, while the symmetric one (which we consider in this paper) is the electromagnetic field potential.

It seems that the analogy which begins with the gravitational law of Newton and Coulomb's law may hopefully be extended to a complete theory in Minkowski space and also to a description of the interaction, gravitational and electromagnetic, in a Riemann space. (We do not dare to talk of a "theory" in the latter case because, in this paper, we have not touched on the issue of field equations.)

Finally, we have calculated the advance of the perihelion of Mercury for this metric tensor. We obtain one third of the result one gets with the Schwarzschild metric. Hence improvement of the theory is here required [for instance along the lines of Majernik (1962a) and Cole (1975)]. However, so far, this description of the gravitational field can be applied to all the experimental results mentioned by Leiby (1972) and by Majernik (1971), since their basic formulas are approximate results of ours, with the same suitable additional hypothesis (which keeps the analogy with electrodynamics).

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